Gravitation as a Super SL(2,C) Gauge Theory

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Abstract

We present a gauge theory of the super SL(2,C) group. The gauge potential is a connection of the Super SL(2,C) group. A MacDowell-Mansouri type of action is proposed where the action is quadratic in the Super SL(2,C) curvature and depends purely on gauge connection. By breaking the symmetry of the Super SL(2,C) topological gauge theory to SL(2,C), a metric is naturally defined.

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Let us start with a Super SL(2,C) algebra[1] (with three complex SL(2,C) generators $M_{00}, M_{01} = M_{10}, M_{11}$ and two complex supersymmetric generators Q_0, Q_1)[2]:

$$[M_{AB}, M_{CD}] = \epsilon_{C(A}M_{B)D} + \epsilon_{D(A}M_{B)C}, \tag{1}$$
$$[M_{AB}, Q_C] = \epsilon_{C(A}Q_{B)}, \qquad \{Q_A, Q_B\} = 2M_{AB}, \tag{2}$$

where $\epsilon_{C(A}M_{B)D} = \frac{1}{2}(\epsilon_{CA}M_{BD} + \epsilon_{CB}M_{AD})$ and $\epsilon_{C(A}Q_{B)} = \frac{1}{2}(\epsilon_{CA}Q_B + \epsilon_{CB}Q_A)$. The Super SL(2,C) group is isomorphic to the complex extension of OSp(1,2). It is a simple super Lie group and has a nondegenerate Killing form [2]. The Cartan-Killing metric is $\eta_{pq} = \mathrm{diag}(\frac{1}{2}(\epsilon_{AM}\epsilon_{BN} + \epsilon_{AN}\epsilon_{BM}), -2\epsilon_{AB})$.

To *gauge* this Super SL(2,C) group[3], we associate to each generator $T_p = \{M_{AB}, Q_A\}$ a 1-form field $A^p = \{\omega^{AB}, \varphi^A\}$, and form a super Lie algebra valued connection 1-form,

$$A = A^pT_p = \omega^{AB}M_{AB} + \varphi^AQ_A, \tag{3}$$

where ω^{AB} is the SL(2,C) connection 1-form and φ^A is an anti-commuting spinor valued 1-form. (We shall use $\mathcal{D}$ for the Super SL(2,C) covariant derivative and D for the SL(2,C) covariant derivative.)

The curvature is given by $F = dA + \frac{1}{2}[A, A]$. Given the Super SL(2,C) connection A defined in equation (3), the curvature ($F = F(M)^{AB}M_{AB} + F(Q)^AQ_A$) contains a bosonic part associated with M_{AB},

$$F(M)^{AB} = d\omega^{AB} + \omega^{AC} \wedge \omega_C{}^B + \varphi^A \wedge \varphi^B; \tag{4}$$

and a fermionic part associated with Q_A ,

$$F(Q)^A = d\varphi^A + \omega^{AB} \wedge \varphi_B. \tag{5}$$

[1]The upper-case Latin letters $A, B, ... = 0, 1$ denote two component spinor indices, which are raised and lowered with the constant symplectic spinors $\epsilon_{AB} = -\epsilon_{BA}$ together with its inverse and their conjugates according to the conventions $\epsilon_{01} = \epsilon^{01} = +1$, $\lambda^A := \epsilon^{AB}\lambda_B$, $\mu_B := \mu^A\epsilon_{AB}$. Lowercase Latin letters $p, q, ...$ denote the Super SL(2,C) group indices, $a, b, c, ... = 0, 1, 2, 3$ denote the SO(3,1) indices [1].

[2]We can realize this Super SL(2,C) algebra by a complex superspace $C^{2|1}$ with coordinates (π_0, π_1, θ) where transformations are given by $M_{AB} = \pi_A\frac{\partial}{\partial\pi^B} + \pi_B\frac{\partial}{\partial\pi^A}$ and $Q_A = \pi_A\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial\pi^A}$ with $\theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\frac{\partial}{\partial\theta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

A simple action, quadratic in the curvature, using this Super SL(2,C) connection A is

$$\mathcal{S}_{\mathrm{T}}[A^p] = \int F^p \wedge F^q \, \eta_{pq} = \int F(M)^{AB} \wedge F(M)_{AB} + 2F(Q)^A \wedge F(Q)_A, \quad (6)$$

where η_{pq} is the Cartan-Killing metric of the Super SL(2,C) group, $\mathcal{D}\eta_{pq} = 0$. However, this action is a total differential. Hence, similar to the work of MacDowell and Mansouri [4], we need to choose another spinor action which is SL(2,C) invariant, thus breaking the topological field theory of the Super SL(2,C) symmetry into an SL(2,C) symmetry. Let us choose $i_{pq} = \mathrm{diag}(\frac{1}{2}\left(\epsilon_{AM}\epsilon_{BN} + \epsilon_{AN}\epsilon_{BM}\right), 0)$ The new action (related to the quadratic spinor action [5, 6, 7, 8, 9, 10]) is

$$\mathcal{S}[A^p] \;\; = \;\; \int F^p \wedge F^q \, i_{pq} = \int F(M)^{AB} \wedge F(M)_{AB}. \quad (7)$$

The field equations are obtained by varying the Lagrangian with respect to gauge potentials (the Super SL(2,C) connection). With these gauge potentials fixed at the boundary, the field equations are

$$R^{AB} \wedge \varphi_B = 0 \qquad (DF(Q)^A = 0), \quad (8)$$

$$D(R^{AB} + \varphi^A \wedge \varphi^B) = 0 \qquad (DF(Q)^{AB} = 0), \quad (9)$$

where, becuase of the SL(2,C) Bianchi identidy ($DR^{AB} = 0$), the second field equation (9) is reduced to $D(\varphi^A \wedge \varphi^B) = 0$.

In order to make a connection between the internal space of the Super SL(2,C) group with the structures on the four-manifold, we break the symmetry[11, 12] from a Super SL(2,C) topological field theory $\mathcal{S}_{\mathrm{T}}[A^p]$ into an SL(2,C) invariant $\mathcal{S}[A^p]$. Using the fact that $\mathcal{D}i_{pq} = C^m{}_{pn}A^n i_{mq}$, and $\mathcal{D}\eta_{pq} = 0$, the metric $\mathcal{G}$ is defined by

$$\mathcal{G} = \eta^{pm}\eta^{qn}\mathcal{D}i_{pq} \otimes \mathcal{D}i_{mn} = \epsilon_{AB}\varphi^A \otimes \varphi^B. \quad (10)$$

Thus upon breaking the supersymmetry, the metric is naturally defined.

The real spacetime metric can be obtained by considering the complex conjugate of the generators $M_{A'B'}, Q_{A'}$ (which satisfies the complex conjugate of (1) and (2)), the gauge potentials (3) and the actions (6), (7). Consequently the spacetime metric is the real part of the complex metric $\mathcal{G}$, and the real tetrad $\theta^{AA'}$ is given by $\varphi^A = \theta^{AA'}Q_{A'}$.

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